Interpreting The Out-Of-Control Signal Of A Dispersion Control Chart

Bartzis Georgios
Department of Statistics and Insurance Science
University of Piraeus
Piraeus, Greece

Contents

Multivariate Statistical Process Control (MSPC)

- In most cases, the products' quality is not related to one but more qualitative characteristics

SO

- It is necessary to monitor more than one characteristics simultaneously for ensuring the total quality of the product
- Also by using independent control charts, the Type I error is falsely determined because the correlation is not taken into account

Harold Hotteling (1947) first applied the idea of MSPC in data regarding bombsights in WWII

ANSWERS OF MSPC

According to Jackson (1991) a multivariate procedure should provide 4 information:

- An answer on whether or not the process is in-control
- An overall probability for the event "procedure diagnoses an out-of –control state erroneously" must be specified
- The relation between the variables-attributes should be taken into account
- An answer to the question "If the process is out-of-control, what is the problem?"

Multivariate Control Charts for the Mean and Dispersion

MSPC for the Mean

For Phase II the most common control chart for monitoring the mean assuming a *p*-dimensional normal distribution for the characteristics of interest, is the X^2 chart with the following form:

$$X_i^2 = n\left(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)'\boldsymbol{\Sigma}_0^{-1}\left(\bar{\mathbf{x}}_i - \boldsymbol{\mu}_0\right)$$

The multivariate Shewhart control chart has only an upper control limit with expression taken from the chi-square distribution. The UCL is computed as follows:

$$UCL = \chi^2_{p,1-\alpha}$$

MSPC for the Dispersion

Due to the fact that in practice the dispersion does not remain constant through time, methodologies have been developed for monitoring the variability of the process.

The monitoring of the dispersion, can be measured by three widely known quantities. These are:

- The determinant of the variance-covariance matrix $|\boldsymbol{\Sigma}|$, which is called the generalized variance and
- The trace of the variance covariance matrix, trace($\boldsymbol{\Sigma}$)
- The Principal Components

Another quantity that can be used for measuring variability is the multivariate Coefficient of Variation

Comparing the Performance of Control Charts

- In most cases the performance of control charts is measured by the Average Run Length (ARL) which is the expected waiting time until the first occurrence of an event creating an out-of-control signal
- The in-control ARL is the average number of plotted samples until an out-of-control signal even though the process is in-control
- The out-of-control ARL is the average number of plotted samples until an out-of-control signal when the process is considered out-of-control

Multivariate Control Charts for the Dispersion

Control Charts

1. Frank Alt (1985) used the unbiased estimator of Σo which is and $\left|\bar{\mathbf{S}}\right| / b_1$ constructed the following chart (CC1):

- $UCL = |\Sigma_0|\left(b_1 + 3\sqrt{b_2}\right)$
- $CL = |\Sigma_0| b_1$
- $LCL = |\Sigma_0|\left(b_1 - 3\sqrt{b_2}\right)$

2. Alt (1985) also proposed the charting of the following statistic:

$$W = -pn + pn\ln n - n\ln\left(\frac{|\mathbf{A}|}{|\Sigma_0|}\right) + trace\left(\Sigma_0^{-1}\mathbf{A}\right)$$

Where the LCL=0 and UCL= $\chi^2_{p(p+1)/2;1-\alpha}$

3. Alt (1985) proposed the charting of $|\mathbf{S}|$ with control limits (CC2):

- $LCL = \dfrac{|\Sigma_0|\left(\chi^2_{2n-4;a/2}\right)^2}{4(n-1)^2}$ and
- $UCL = \dfrac{|\Sigma_0|\left(\chi^2_{2n-4;1-a/2}\right)^2}{4(n-1)^2}$

4. For the $|\mathbf{S}|^{1/2}$ Control Chart, Alt computed the following control limits (CC3):

- $UCL = |\Sigma_0|^{1/2}\left(b_3 + 3\sqrt{b_1 - b_3^2}\right)$

- $LCL = |\Sigma_0|^{1/2}\left(b_3 - 3\sqrt{b_1 - b_3^2}\right)$ and $CL = |\Sigma_0|^{1/2} b_3$

6. Quinino et al. (2012) introduced the VMIX statistic

$$VMIX = \frac{\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2}{2n}$$

and the process is considered out-of-control if VMIX>UCL which is computed for a predetermined ARL

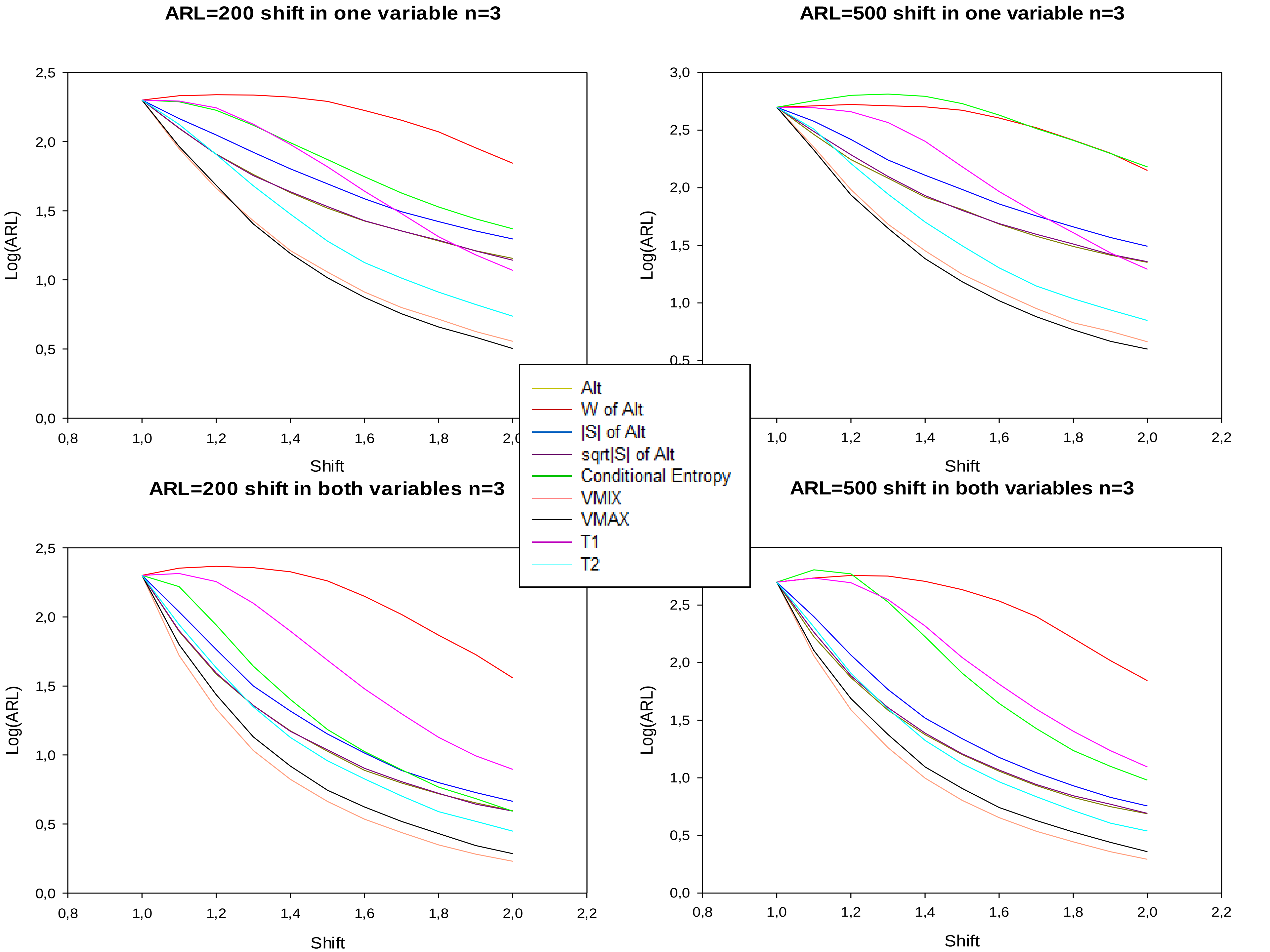
6. Machado and Costa (2008) proposed an EWMA scheme based on the statistic $Z_i = \lambda Y_i + (1-\lambda)Z_{i-1}$ where $Y_i = \max\{S_x^2, S_y^2\}$ and the chart signals for if Z>UCL

7. Hung and Chen (2012) applied the Cholesky Decomposition on the variance-covariance matrix and created two Statistics (T1, T2) for monitoring the variability in a multivariate process

Control Chart Comparison

- Regarding the comparison of the various Control Charts, the control limits of the charts were computed for achieving an in-control ARL=200 and 500.
- Scenarios for different sample sizes have been taken into account (n= 3, 5, 10, 50).
- The out-of-control ARL is compared for a shift in one or both variances (shift by $k\sigma^2$ with k=1, 1.1, 1.2,…, 2).
- Also, the shift considered to be in either one or both variables.

The Comparison


ARL=200 shift in one variable n=3
ARL=500 shift in one variable n=3
ARL=200 shift in both variables n=3
ARL=500 shift in both variables n=3
Log(ARL)
Shift
0,0
0,5
1,0
1,5
2,0
2,5
3,0
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
Alt
W of Alt
|S| of Alt
sqrt|S| of Alt
Conditional Entropy
VMIX
VMAX
T1
T2

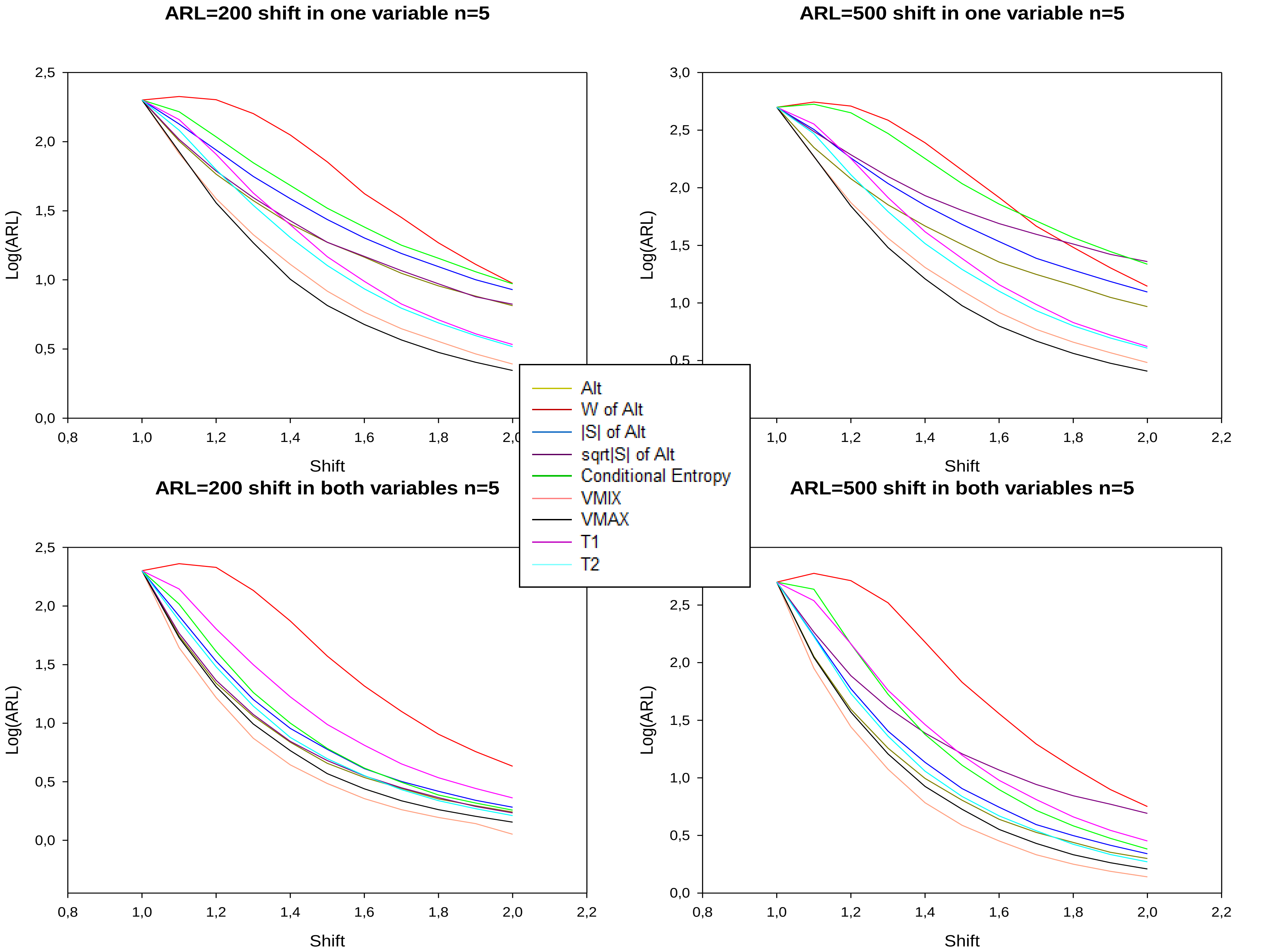
Sample Size (3)

In this case it seems that the VMAX chart has the best performance for shift only in one variable regardless the in-control ARL.

The VMIX chart seems to have the best performance in a shift in both variables regardless the in-control ARL

SO both bivariate charts have the best performance

From the multivariate charts, the T2 chart has the best performance for shifts bigger than. The sqrt|**S**| and Alts' chart with unbiased estimator $|\bar{S}|/b_1$ perform better for shifts smaller than 1.2 σ^2.



ARL=200 shift in one variable n=5
ARL=500 shift in one variable n=5
ARL=200 shift in both variables n=5
ARL=500 shift in both variables n=5
Log(ARL)
Shift
0,0
0,5
1,0
1,5
2,0
2,5
3,0
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
Alt
W of Alt
|S| of Alt
sqrt|S| of Alt
Conditional Entropy
VMIX
VMAX
T1
T2

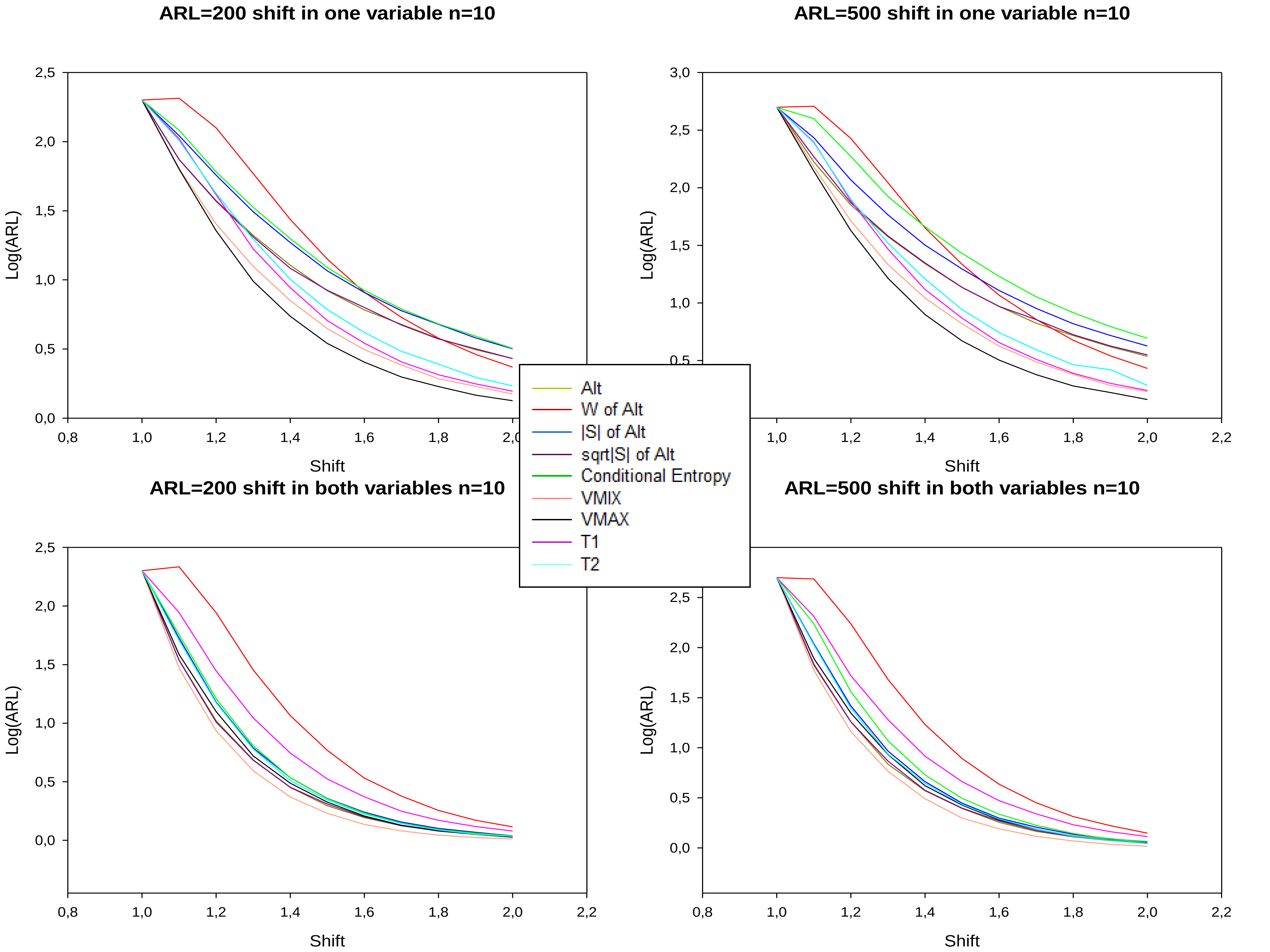
Sample Size (5)

Also in this case the VMAX chart has the best performance for shift only in one variable regardless the in-control ARL and the VMIX chart has the best performance for shifts in both variables regardless the in-control ARL

AGAIN both bivariate charts have the best performance

For the multivariate charts:

- For shifts in one variable, T2 chart has better performance for a shift bigger than 1.3. Otherwise, the sqrt**|S|** and Alts' chart with unbiased estimator $|\bar{S}|/b_1$ perform better.
- For shifts in both variables, sqrt**|S|** and Alts' chart with unbiased estimator $|\bar{S}|/b_1$ perform better for shifts smaller than 1.7 σ^2. Otherwise, T2 performs better.



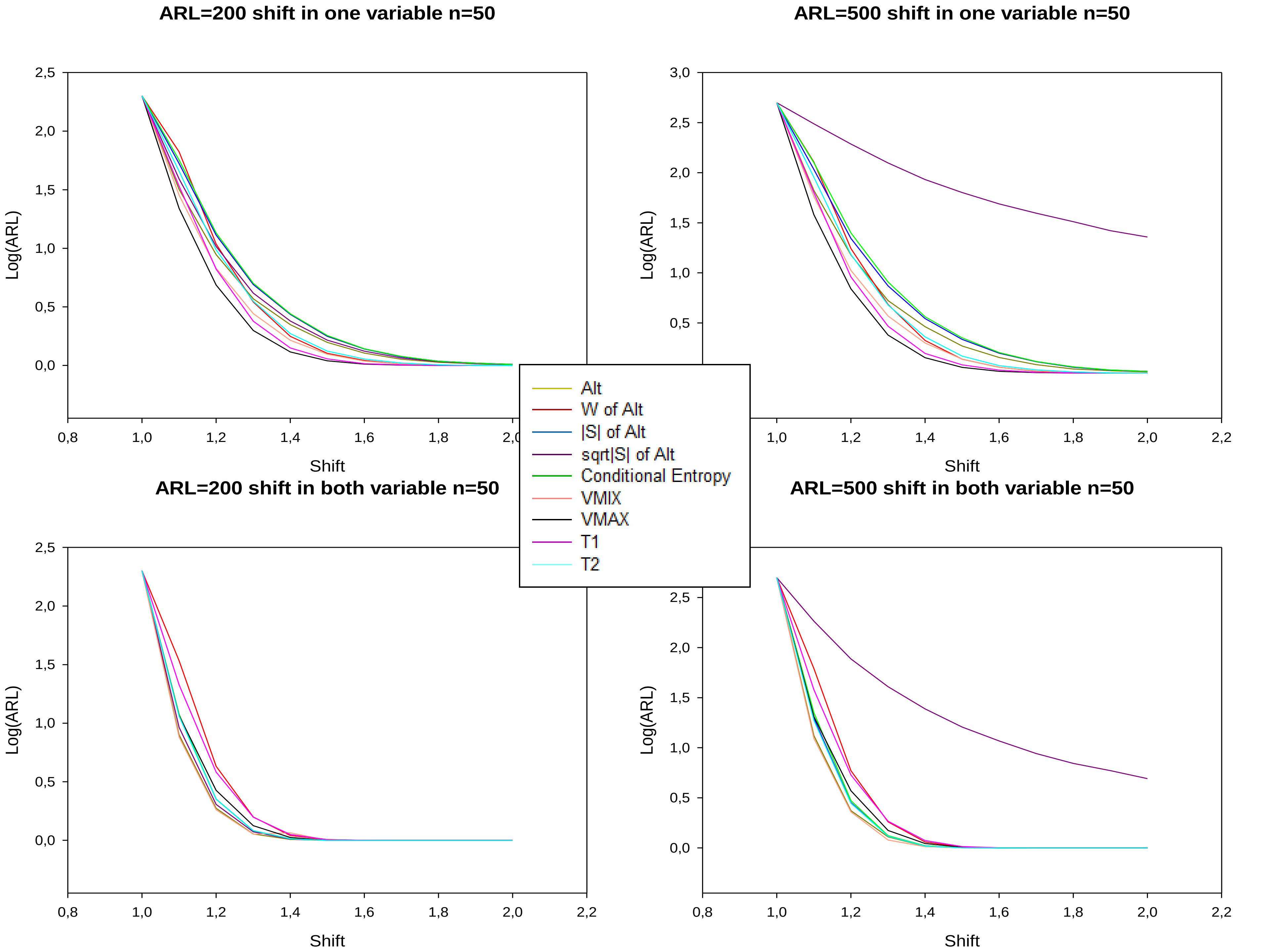
ARL=200 shift in one variable n=10
ARL=500 shift in one variable n=10
ARL=200 shift in both variables n=10
ARL=500 shift in both variables n=10
Log(ARL)
Shift
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
0,0
0,5
1,0
1,5
2,0
2,5
3,0
Alt
W of Alt
|S| of Alt
sqrt|S| of Alt
Conditional Entropy
VMIX
VMAX
T1
T2

Sample Size (10)

For sample size equal to 10, we have the same pattern for the bivariate charts.

From the multivariate charts, T2 and T1 seem to perform better for shifts in just one variable.

For shifts in both variables, all charts seem to perform almost the same except Alts' W and T1

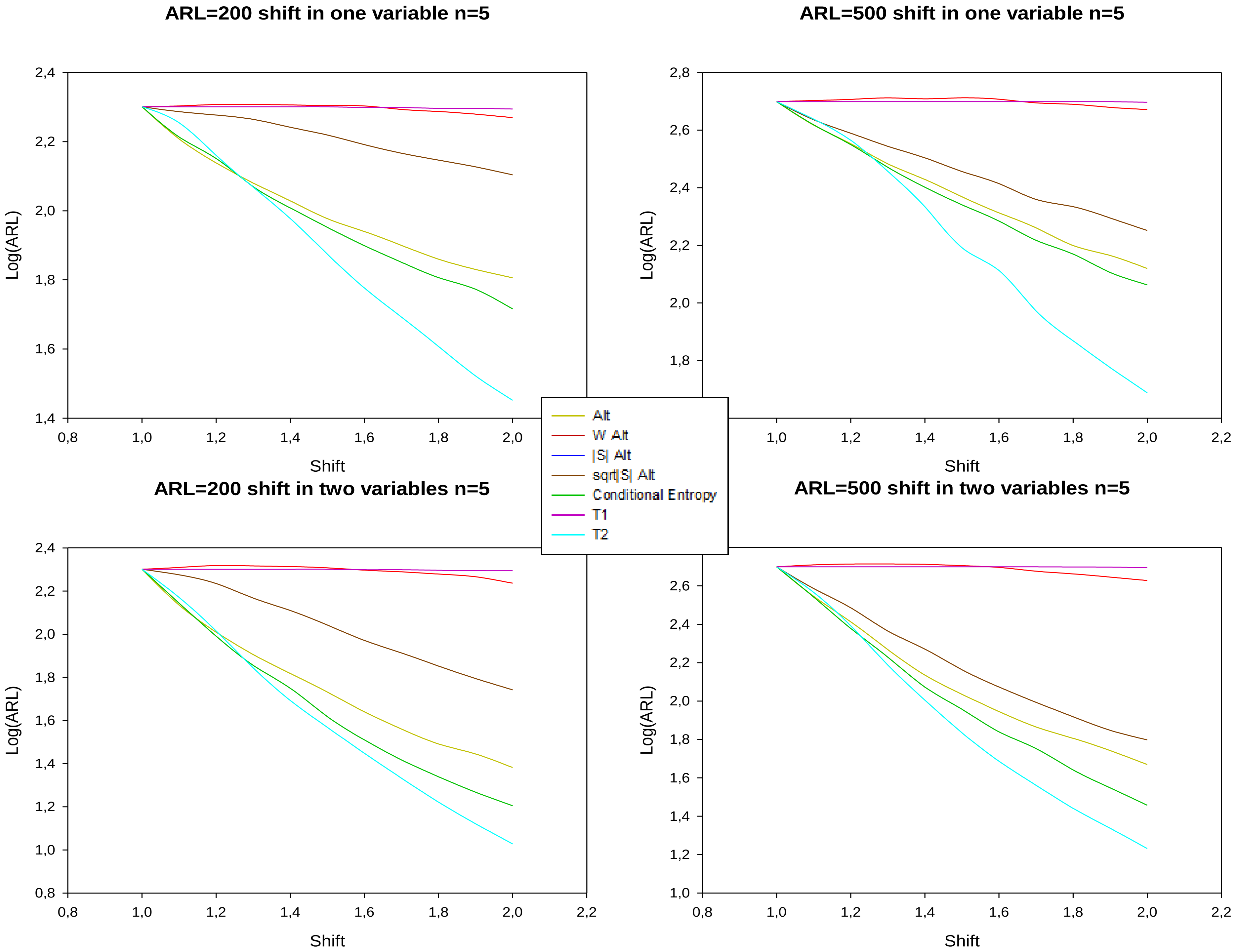

ARL=200 shift in one variable n=50
ARL=500 shift in one variable n=50
ARL=200 shift in both variable n=50
ARL=500 shift in both variable n=50
Log(ARL)
Shift
Alt
W of Alt
|S| of Alt
sqrt|S| of Alt
Conditional Entropy
VMIX
VMAX
T1
T2

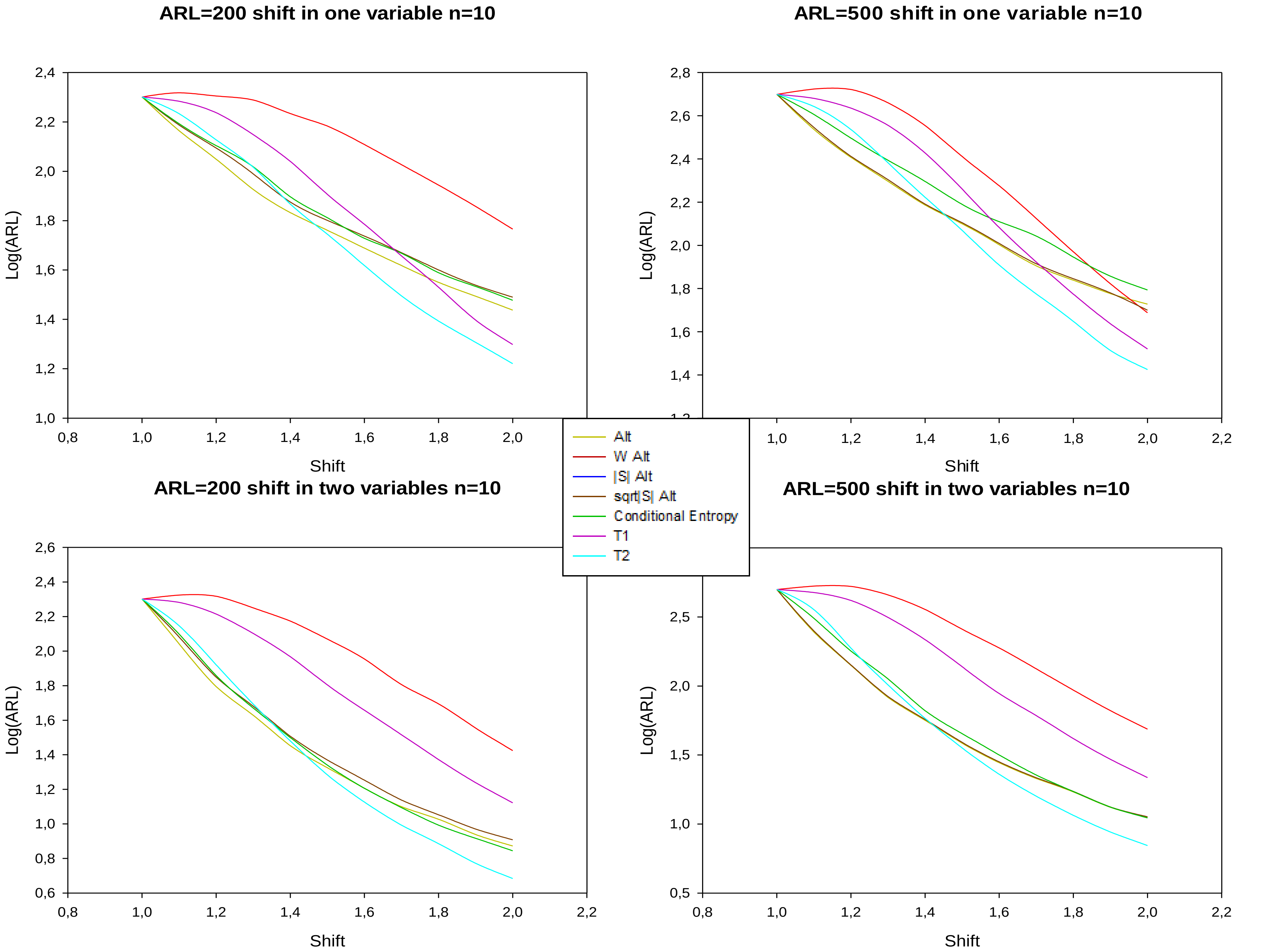
Sample Size (50)

For sample size equal to 50, all charts seem to perform similarly.

Only the sqrt**|S|** seem to have the worst performance only for in-control ARL=500 whether or not the shift occurs in one or both variables.

The Multivariate Case (p=4)

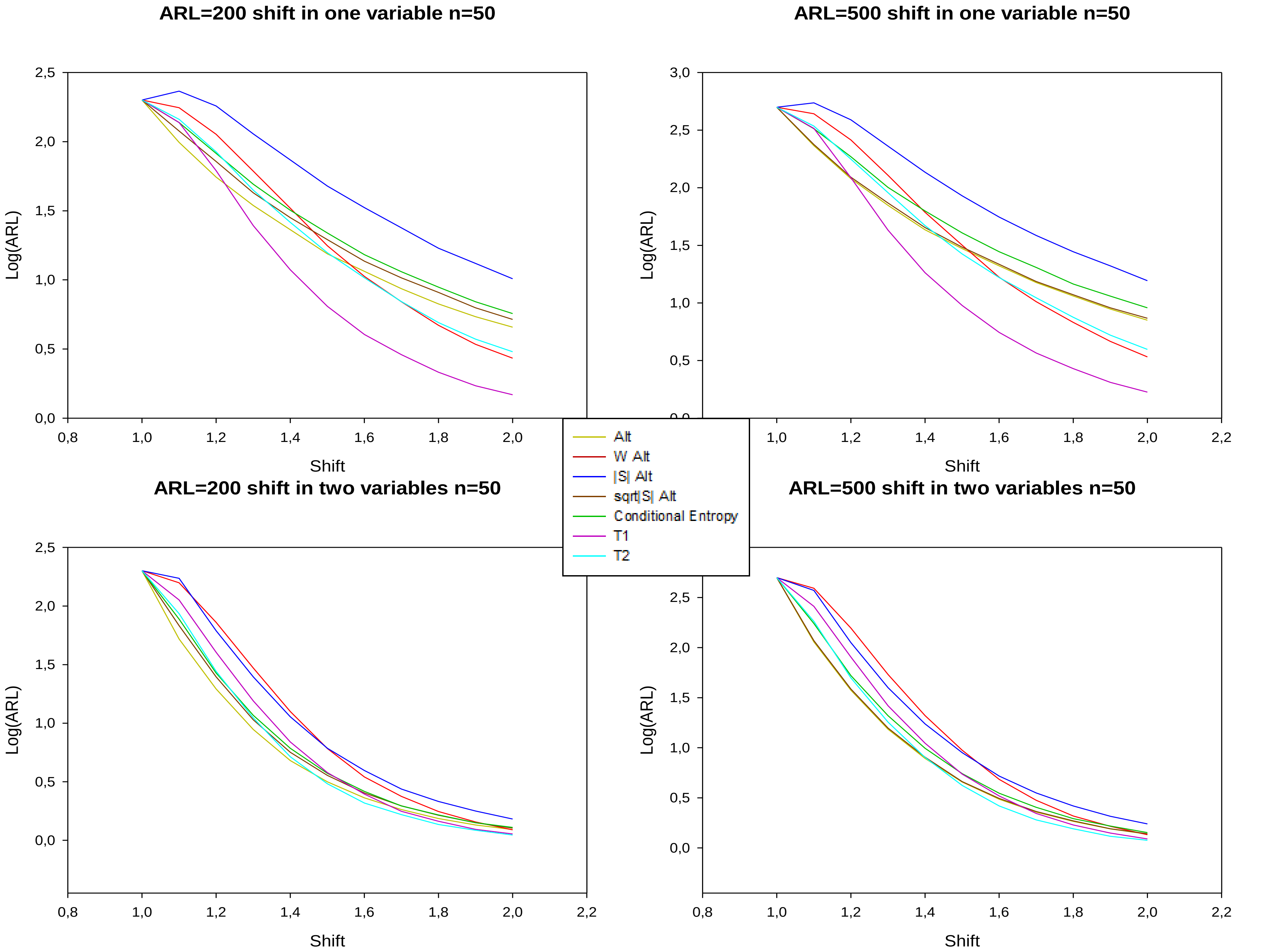

ARL=200 shift in one variable n=5
ARL=500 shift in one variable n=5
ARL=200 shift in two variables n=5
ARL=500 shift in two variables n=5
Log(ARL)
Shift
Alt
W Alt
|S| Alt
sqrt|S| Alt
Conditional Entropy
T1
T2
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
2,4
2,6
2,8



ARL=200 shift in one variable n=10
ARL=500 shift in one variable n=10
ARL=200 shift in two variables n=10
ARL=500 shift in two variables n=10
Log(ARL)
Shift
Alt
W Alt
|S| Alt
sqrt|S| Alt
Conditional Entropy
T1
T2

Sample Size (5, 10)

In all cases shown, the T2 chart has a better performance in big shifts but the chart based on the conditional entropy and the chart with unbiased estimator $|\bar{\mathbf{s}}|/b_1$ perform better for small shifts.


ARL=200 shift in one variable n=50
ARL=500 shift in one variable n=50
ARL=200 shift in two variables n=50
ARL=500 shift in two variables n=50
Log(ARL)
Shift
0,0
0,5
1,0
1,5
2,0
2,5
3,0
0,8
1,0
1,2
1,4
1,6
1,8
2,0
2,2
Alt
W Alt
|S| Alt
sqrt|S| Alt
Conditional Entropy
T1
T2

Sample Size (50)

In the case were a shift occurs in one variable, the T1 chart seems to perform better for big shifts. For small shifts the chart with unbiased estimator $|\bar{S}|/b_1$ and sqrt$|\mathbf{S}|$ has better performance

For shift in both variables, all charts seem to perform the same except $|\mathbf{S}|$and Alts' W

Conclusions

What we saw

- It is clear that the VMAX chart by Machado and Costa (2008) and the VMIX chart by Quinino (2012) perform better in all bivariate cases.

Both charts are used for a bivariate process only and it is clear that in these cases they should be preferred

What we saw

- From the remaining control charts, T2 statistic developed by Hung and Chen (2012) performs really good in large shifts either in one or both variables regardless the sample size
- T1 statistic also developed by Hung and Chen (2012) performs better for a shift in one variable and also for big sample size

What we saw

- Alts' (1985) control chart performs good for large shifts in both variables
- Alts' sqrt|S| seems to perform well in small shifts either in one or both variables

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Thank You For Your Attention

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